The Concentration and Dispersion Coefficient of the Advection Dispersion Equation

Tieliang Wang[1], Kan Zhang[*2], Xue Feng[3]

[1,2]Postdoctoral of Agricultural Engineering, Shenyang Agricultural Uinversity, Shenyang, 110866, China

[2,3]College of Sciences, Shenyang Agricultural University, Shenyang 110866, China

[1]wangzhangpaper@163.com; [*2]kanzhang2004@163.com; [3]xfeng2000@163.com

Abstract

In this paper, some recent developments in the analysis of the BMO space have been applied to the advection dispersion equation. Then the upper and lower bound have been obtained for the concentration and dispersion coefficient of the advection dispersion equation under the conditions where source term and stirring term of advection dispersion equation belong to BMO space.

Keywords

Advection Dispersion Equation; BMO; Dispersion Coefficient

Introduction

Advection dispersion equation is a kind of basic equations, which can be used for convection diffusion problems. For the passive scalar, complicated behavior is often observed even for laminar velocity fields. This is the well-known effect of chaotic advection in (Aref 1984; Ottino 1989). Thus the source and stirring term of the advection dispersion equation can be selected to be any divergence-free, possibly time-dependent flow field. The upper and lower bound of concentration and dispersion coefficient play an important role in comparing the relative effectiveness of various scenarios and analyzing adsorption and desorption of some elements in research area, so effort is focused on researching the concentration and dispersion coefficient of advection dispersion equation in (Griecoa 2005; Shawa 2007). It has been confirmed that the location of the bounding parabola depends on the structure of stirring term and the source term in (Plasting 2006). Bounds on the derivative moments of the concentration field in (Schumacher 2003) have been produced, as well as the bound on mixing efficiency for the advection dispersion equation in (Thiffeault 2004).

The classical BMO space was introduced by John-Nirenberg in (John 1961; Garnett 1981). A new type of BMO, denoted by $BMO_{\partial}(\Omega)$, has been introduced in (Bekolle 1990; Berger 1998) for any bounded domain Ω of the complex space C^n. In this paper, the BMO space is defined by the Bergman metric. For $1 \le p < \infty$, we write $\|\cdot\|_{\alpha,p}$ for the norm on $L^p(B_n, dv_\alpha)$ and $\langle \cdot,\cdot \rangle_\alpha$ for the inner product on $L^2(B_n, dv_\alpha)$, where B_n is the unit ball in C^n and v_α is the Lebesgue volume measure on B_n. For $z \in B_n$, ψ_z is the automorphism of B_n such that $\psi_z(0) = z$ and $\psi_z = \psi_z^{-1}$, which is described in (Rudin 1980). The Bergman space $A_\alpha^2(B_n)$ is the space of holomorphic functions which are square-integrable with respect to measuring dv_α on B_n. Reproducing kernels K_ω^α and normalized reproducing kernels k_ω^α in $A_\alpha^2(B_n)$ are given by, respectively,

$$K_\omega^\alpha(z) = \frac{1}{(1-\langle z,\omega \rangle)^{n+\alpha+1}}$$

$$k_\omega^\alpha(z) = \frac{(1-|\omega|^2)^{\frac{n+\alpha+1}{2}}}{(1-\langle z,\omega \rangle)^{n+\alpha+1}}$$

for $z, \omega \in B_n$.

For $f \in L^1(B_n, dv_\alpha)$, the Berezin transform of f is defined as the function $\tilde{f}$, that is

$$\tilde{f}(z) = \int_{B_n} f(\omega) \left| k_z^\alpha(\omega) \right|^2 dv_\alpha(\omega)$$

Let $f \in L^1(B_n, dv_\alpha)$ an $p \ge 1$, it is said that $f \in BMO_\alpha^p(B_n)$ whenever

$$\|f\|_{BMO_\alpha^p} = \sup_{z \in B_n} \left\| f \circ \psi_z - \tilde{f}(z) \right\|_{\alpha,p} < \infty$$

By the Theorem 5 in (Zhu 1992), the fact is well known that $BMO_\alpha^p(B_n)$ is equivalent to $BMO_r^p(B_n)$ in (Zhu 1992).

For $z, \omega \in B_n$, let

$$\beta(z,\omega) = \frac{1}{2}\log\frac{1+|\psi_z(\omega)|}{1-|\psi_z(\omega)|}$$

denote the Bergman metric on B_n. For any $z \in B_n$ and $r > 0$, let

$$D(z) = \{\omega \in B_n | \beta(z,\omega) < r\}$$

be the Bergman metric ball with center z and radius r. Let $|D(z)|$ denote the volume of $D(z)$, i.e. $|D(z)| = v_\alpha(D(z))$.

For $f \in L^1(B_n, dv_\alpha)$, the average of f over $D(z)$ is defined by

$$\hat{f}(z) = \frac{1}{|D(z)|}\int_{D(z)} f(\omega) dv_\alpha(\omega)$$

Using the properties of Bergman metric, it follows that

$$\sup_{z\in B_n} = \frac{1}{|D(z)|}\int_{D(z)} \left| f(\omega) - \hat{f}(z)\right|^p dv_\alpha(\omega) < \infty$$

if and only if $\|f\|_{BMO_\alpha^p} < \infty$.

So the functions in $BMO_\alpha^p(B_n)$ have bounded mean oscillation in the Bergman metric and the BMO space plays a significant role in operator theory and applied science, for example (Duraiswamy 2009; Zhang 2011). Since $BMO_\alpha^p(B_n)$ functions are locally in $L^p(B_n, dv_\alpha)$, it is easy to see that

$$L^\infty(B_n) \subseteq BMO_\alpha^p(B_n) \subseteq L^p(B_n, dv_\alpha), \quad p \ge 1$$

$$BMO_\alpha^q(B_n) \subseteq BMO_\alpha^p(B_n) \subseteq BMO_\alpha^1(B_n), \quad 1 \le p \le q$$

From now on we will write BMO_α^p instead of $BMO_\alpha^p(B_n)$.

In this paper, some recent developments in the BMO space have been applied to the advection dispersion equation. Then the upper and lower bound have been obtained for the concentration and dispersion coefficient of the advection dispersion equation under the conditions where source term and stirring term of advection dispersion equation belong to BMO space, further the upper and lower bound of the equivalent dispersion coefficient are acquired.

Throughout the paper, the letter c is utilized to denote a generic positive constant that can change its value at each occurrence.

System Description

The advection dispersion equation for the concentration $\theta(x,t)$ of a passive scalar is

$$\frac{\partial\theta}{\partial t} + D\cdot\nabla\theta = k\Delta\theta + s_0^d + s_t^d \tag{2.1}$$

Where T = top width of the channel; A = wetted cross-sectional area; C = area's average dissolved heavy metal; Q_L = lateral inflow or outflow discharge; C_L = lateral inflow or outflow dissolved heavy metal concentration; Δx = distance between two consecutive cross sections which can be either constant or variable; H = averaged depth over the cross section; U_* =local shear velocity; k =dispersion coefficient, $k > 0$; $u(x,t)$ = cross-section average velocity;

$$D = [7.428 + 1.775(\frac{\mathrm{T}}{\mathrm{H}})^{0.620}(\frac{\mathrm{U}_*}{\mathrm{u}})^{0.572}]Hu(\frac{u}{\mathrm{U}_*})$$

$S_0^d = \frac{Q_L}{C_l}\Delta x$; $S_t^d = kCA$; Q, T, H, U_* are constants.

Without loss of generality, it may be supposed that S_0^d, S_t^d, $\theta(x,t)$ have zero spatial mean at all the times and belong to BMO space, which is true under the realistic condition. Then research domain is divided in which finite unit ball B_n is taken into consideration and at the same time, it is assumed that both the source term and stirring term act on a comparable scale $1 \le l \le L$,where L is length of side of research area and $\frac{L}{l}$ is an integer. The research time is supposed to be $\frac{t}{\tau}$ on the unit ball, where τ is an appropriate time scale characterizing the source and stirring term and $0 \le t \le 1$. For $0 \le x \le l$, $S_0^d(\frac{x}{l},\frac{t}{\tau})$, $S_t^d(\frac{x}{l},\frac{t}{\tau})$ and $u(\frac{x}{l},\frac{t}{\tau})$ are the source term and stirring term on the unit ball. In order to keep track of the effects of the amplitudes of the source term and stirring term on the upper and lower bound for the concentration $\|\theta\|_{BMO_\alpha^2}$ and dispersion coefficient k, supposing that $\|s_0^d + s_t^d\|_{BMO_\alpha^2} = S$ and $\|u\|_{BMO_\alpha^2} = U$.

Suppose the advection dispersion operator defined by

$$L = \frac{\partial}{\partial t} + D\cdot\nabla - k\Delta$$

then the adjoint of the advection dispersion operator is

$$L^* = \frac{\partial}{\partial t} - D \cdot \nabla - k\Delta$$

Next, the upper and lower bound of

$\|\theta\|_{BMO_\alpha^2}$ will be obtained for given S and U.

Since

$$S = \|s_0^d + s_t^d\|_{BMO_\alpha^2} \le \|L\|_{BMO_\alpha^2}\|\theta\|_{BMO_\alpha^2} = \|L^*\|_{BMO_\alpha^2}\|\theta\|_{BMO_\alpha^2}$$

then we have

$$S \le \sup_{\|\psi\|_{BMO_\alpha^2}=1} [\|-\frac{\partial \psi}{\partial t} - D\cdot\nabla\psi - k\Delta\psi\|_{BMO_\alpha^2}]\|\theta\|_{BMO_\alpha^2} \quad (2.2)$$

Where ψ is an arbitrary smooth noralized spatially periodic function.

By the application of the Minkowski inequality and Cauchy inequality to formula (2.2), it can be observed that

$$\begin{aligned} S &\le [\sup_{\|\psi\|_{BMO_\alpha^2}=1}\left\|\frac{\partial\psi}{\partial t}\right\|_{BMO_\alpha^2} + k\sup_{\|\psi\|_{BMO_\alpha^2}=1}\|\Delta\psi\|_{BMO_\alpha^2} \\ &\quad + \|D\|_{BMO_\alpha^2}\sup_{\|\psi\|_{BMO_\alpha^2}=1}\|\nabla\psi\|_{BMO_\alpha^2}]\|\theta\|_{BMO_\alpha^2} \\ &\le [\sup_{\|\psi\|_{BMO_\alpha^2}=1}\left\|\frac{\partial\psi}{\partial t}\right\|_{BMO_\alpha^2} + k\sup_{\|\psi\|_{BMO_\alpha^2}=1}\|\Delta\psi\|_{BMO_\alpha^2} \\ &\quad + \frac{7.428H\|u\|_{BMO_\alpha^2}^2}{U_*}\sup_{\|\psi\|_{BMO_\alpha^2}=1}\|\nabla\psi\|_{BMO_\alpha^2} \\ &\quad + \frac{1.775H^{0.38}T^{0.62}\|u\|_{BMO_\alpha^2}^{1.428}}{(U_*)^{0.428}}\sup_{\|\psi\|_{BMO_\alpha^2}=1}\|\nabla\psi\|_{BMO_\alpha^2}]\|\theta\|_{BMO_\alpha^2} \\ &= c\|u\|_{BMO_\alpha^2}\|\theta\|_{BMO_\alpha^2} \\ &= cU\|\theta\|_{BMO_\alpha^2} \end{aligned} \quad (2.3)$$

Where

$$\begin{aligned} c &= \frac{\sup_{\|\psi\|_{BMO_\alpha^2}=1}\left\|\frac{\partial\psi}{\partial t}\right\|_{BMO_\alpha^2} + k\sup_{\|\psi\|_{BMO_\alpha^2}=1}\|\Delta\psi\|_{BMO_\alpha^2}}{\|u\|_{BMO_\alpha^2}} \\ &\quad + \frac{7.428H\|u\|_{BMO_\alpha^2}}{U_*}\sup_{\|\psi\|_{BMO_\alpha^2}=1}\|\nabla\psi\|_{BMO_\alpha^2} \\ &\quad + \frac{1.775H^{0.38}T^{0.620}\|u\|_{BMO_\alpha^2}^{0.428}}{(U_*)^{0.428}}\sup_{\|\psi\|_{BMO_\alpha^2}=1}\|\nabla\psi\|_{BMO_\alpha^2} \end{aligned} \quad (2.4)$$

By the formula (2.3), it is clear that

$$\|\theta\|_{BMO_\alpha^2} \ge \frac{S}{cU} \quad (2.5)$$

By the formula (2.5), it is clear that we have the freedom to choose ψ to optimize the lower bound of the concentration for the particular source and stirring term. From formula (2.5), the fact is learnt that increasing U at fixed S should decrease the lower bound of concentration. While increasing S at fixed U should augment the lower bound. Conversely, the increasing of concentration does not imply the increment of source amplitude or stirring amplitude.

However, the concentration $\|\theta\|_{BMO_\alpha^2}$ with large amplitude almost produces the large source amplitude S unless U is decreased sufficiently. It will be proved that the lower bound of the concentration is proportional to $\frac{Sl}{U}$, so that a source with large fluctuations necessarily produces a poorly mixed state unless U is increased sufficiently.

By formula (2.4), (2.5) and substituting the scaled variables $T = \frac{t}{\tau}$ and $y = \frac{x}{l}$, it is easy to prove that

$$\|\theta\|_{BMO_\alpha^2} \ge \frac{Sl}{U[S_r\sup_{\|\psi\|_{BMO_\alpha^2}=1}\left\|\frac{\partial\psi}{\partial T}\right\|_{BMO_\alpha^2} + P_e^{-1}\sup_{\|\psi\|_{BMO_\alpha^2}=1}\|\Delta_y\psi\|_{BMO_\alpha^2} + c_1]} \quad (2.6)$$

Where

$$\begin{aligned} c_1 &= \frac{7.428H}{U_*}\sup_{\|\psi\|_{BMO_\alpha^2}=1}\|\nabla\psi\|_{BMO_\alpha^2} \\ &\quad + \frac{1.775H^{0.38}T^{0.620}}{(U_*)^{0.428}\|u\|_{BMO_\alpha^2}^{0.572}}\sup_{\|\psi\|_{BMO_\alpha^2}=1}\|\nabla\psi\|_{BMO_\alpha^2} \end{aligned} \quad (2.7)$$

Here $P_e = \frac{Ul}{k}$ is the Peclet number and $S_r = \frac{l}{U\tau}$ is the Strouhal number.

If it possible to choose ψ to be time-independent and still satisfy formula (2.6) and (2.7), then we get the lower bound of concentration

$$\|\theta\|_{BMO_\alpha^2} \ge \frac{Sl}{U[P_e^{-1}\sup_{\|\psi\|_{BMO_\alpha^2}=1}\|\Delta_y\psi\|_{BMO_\alpha^2} + c_1]} \quad (2.8)$$

By the formula (2.6) and the assumption that ψ is time-dependent, then the lower of concentration is affected by Peclet number and Strouhal number, holding the other parameter constant. By formula (2.8) and the assumption that ψ is time-independent, then the lower of concentration is only affected by Peclet number, holding the other parameter constant. As known, the concentration and dispersion coefficient can be affected by several environmental and biochemical factors such as temperature, PH, EC, salinity, etc (Nassehi 1993; Kashefipour 2002; Balmforth 2003; Roshanfekr 2008). In fact, formula (2.6) and (2.8) also imply that there must have lots of complex factors influencing the concentration and transfer theory.

By the formula (2.4) and (2.5), we get the lower bound of dispersion coefficient k.

$$k \ge \frac{S}{\|\theta\|_{BMO_\alpha^2} \sup\limits_{\|\psi\|_{BMO_\alpha^2}=1} \|\Delta\psi\|_{BMO_\alpha^2}} - \frac{\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1} \left\|\frac{\partial\psi}{\partial t}\right\|_{BMO_\alpha^2}}{\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1} \|\Delta\psi\|_{BMO_\alpha^2}}$$
$$-\frac{7.428H\|u\|^2_{BMO_\alpha^2}}{\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1} \|\Delta\psi\|_{BMO_\alpha^2} U_*} \sup\limits_{\|\psi\|_{BMO_\alpha^2}=1} \|\nabla\psi\|_{BMO_\alpha^2} \tag{2.9}$$
$$-\frac{1.775H^{0.38}T^{0.620}\|u\|^{1.428}_{BMO_\alpha^2}}{\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1} \|\Delta\psi\|_{BMO_\alpha^2} (U_*)^{0.428}} \sup\limits_{\|\psi\|_{BMO_\alpha^2}=1} \|\nabla\psi\|_{BMO_\alpha^2}.$$

If ψ is time-independent and satisfies the formula (2.9), then we have

$$k \ge \frac{S}{\|\theta\|_{BMO_\alpha^2} \sup\limits_{\|\psi\|_{BMO_\alpha^2}=1} \|\Delta\psi\|_{BMO_\alpha^2}}$$
$$-\frac{7.428H\|u\|^2_{BMO_\alpha^2}}{\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1} \|\Delta\psi\|_{BMO_\alpha^2} U_*} \sup\limits_{\|\psi\|_{BMO_\alpha^2}=1} \|\nabla\psi\|_{BMO_\alpha^2}$$
$$-\frac{1.775H^{0.38}T^{0.620}\|u\|^{1.428}_{BMO_\alpha^2}}{\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1} \|\Delta\psi\|_{BMO_\alpha^2} (U_*)^{0.428}} \sup\limits_{\|\psi\|_{BMO_\alpha^2}=1} \|\nabla\psi\|_{BMO_\alpha^2}$$

Next, we will obtain the upper bound for $\|\theta\|_{BMO_\alpha^2}$. Using poincare's inequality, Frechet-Jordan-Von Neumann theorem and the fact that

$$x\cdot y = \frac{1}{4}(\|x+y\|^2_{BMO_\alpha^2} - \|x-y\|^2_{BMO_\alpha^2})$$

for $x, y \in BMO_\alpha^2$, it is clear that

$$\begin{aligned} &\left\|s_0^d + s_t^d - \frac{\partial\theta}{\partial t} + k\Delta\theta\right\|_{BMO_\alpha^2} \\ &= \|D\cdot\nabla\theta\|_{BMO_\alpha^2} \\ &\ge \frac{1}{2}(\|D\|^2_{BMO_\alpha^2} + \|\nabla\theta\|^2_{BMO_\alpha^2} - \|D-\nabla\theta\|^2_{BMO_\alpha^2}) \\ &\ge \frac{1}{2}(\|D\|^2_{BMO_\alpha^2} + d\|\theta\|^2_{BMO_\alpha^2} - \|D-\nabla\theta\|^2_{BMO_\alpha^2}) \end{aligned} \tag{2.10}$$

where d is poincare's number.

By the formula (2.10), it is easy to prove

$$\begin{aligned} &\|D\|^2_{BMO_\alpha^2} + d\|\theta\|^2_{BMO_\alpha^2} \\ &\le 2\left\|s_0^d + s_t^d - \frac{\partial\theta}{\partial t} + k\Delta\theta\right\|_{BMO_\alpha^2} + \|D-\nabla\theta\|^2_{BMO_\alpha^2} \\ &\le 2S + 2\left\|\frac{\partial\theta}{\partial t}\right\|_{BMO_\alpha^2} + 2k\|\Delta\theta\|_{BMO_\alpha^2} + \|D\|^2_{BMO_\alpha^2} \\ &+ 2\|D\|_{BMO_\alpha^2}\|\nabla\theta\|_{BMO_\alpha^2} + \|\nabla\theta\|^2_{BMO_\alpha^2} \end{aligned} \tag{2.11}$$

By formula (2.11), it is obtained that

$$\begin{aligned} &c\|\theta\|^2_{BMO_\alpha^2} \\ &\le 2S + 2\left\|\frac{\partial\theta}{\partial t}\right\|_{BMO_\alpha^2} + 2k\|\Delta\theta\|_{BMO_\alpha^2} + 2\|D\|_{BMO_\alpha^2}\|\nabla\theta\|_{BMO_\alpha^2} + \|\nabla\theta\|^2_{BMO_\alpha^2} \\ &\le 2S + 2\left\|\frac{\partial\theta}{\partial t}\right\|_{BMO_\alpha^2} + 2k\|\Delta\theta\|_{BMO_\alpha^2} + \|\nabla\theta\|^2_{BMO_\alpha^2} + 2(\frac{7.428H\|u\|^2_{BMO_\alpha^2}}{U_*} + \\ &1.775T^{0.62}H^{0.38}U_*^{-0.428}\|u\|^{1.428}_{BMO_\alpha^2})\|\nabla\theta\|_{BMO_\alpha^2} \\ &\le 2S + 2\left\|\frac{\partial\theta}{\partial t}\right\|_{BMO_\alpha^2} + 2k\|\Delta\theta\|_{BMO_\alpha^2} + \|\nabla\theta\|^2_{BMO_\alpha^2} + (\frac{14.856H\|u\|^2_{BMO_\alpha^2}}{U_*} + \\ &3.55T^{0.62}H^{0.38}U_*^{-0.428}\|u\|^{1.428}_{BMO_\alpha^2})\|\nabla\theta\|_{BMO_\alpha^2}. \end{aligned} \tag{2.12}$$

Let

$$\varepsilon_1 = \left\|\frac{\partial\theta}{\partial t}\right\|_{BMO_\alpha^2}, \varepsilon_2 = \|\Delta\theta\|_{BMO_\alpha^2}, \varepsilon_3 = \|\nabla\theta\|_{BMO_\alpha^2}$$

By formula (2.12), we obtain

$$\|\theta\|_{BMO_\alpha^2} \le c\sqrt{2S + 2\varepsilon_1 + 2k\varepsilon_2 + (c_3U^{0.572} + c_4)U^{1.428}\varepsilon_3 + \varepsilon_3^2} \tag{2.13}$$

$$c_3 = 14.856\frac{H}{U_*}, c_4 = 3.55\frac{H^{0.38}T^{0.620}}{(U_*)^{0.428}}$$

If θ is time-independent and still satisfies formula (2.13), then we acquire

$$\|\theta\|_{BMO_\alpha^2} \le c\sqrt{2S + 2k\varepsilon_2 + (c_3U^{0.572} + c_4)U^{1.428}\varepsilon_3 + \varepsilon_3^2} \quad (2.14)$$

$$c_3 = 14.856\frac{H}{U_*}, c_4 = 3.55\frac{H^{0.38}T^{0.620}}{(U_*)^{0.428}}.$$

By the formula (2.5) and (2.13), we get the upper and lower bound for the concentration $\|\theta\|_{BMO_\alpha^2}$.

$$\frac{S}{cU} \le \|\theta\|_{BMO_\alpha^2} \le c\sqrt{2S + 2\varepsilon_1 + 2k\varepsilon_2 + (c_3U^{0.572} + c_4)U^{1.428}\varepsilon_3 + \varepsilon_3^2} \quad (2.15)$$

If θ is time-independent and still satisfies formula (2.8) and (2.14), then we obtain

$$\frac{Sl}{U[P_e^{-1}\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1}\|\Delta_y\psi\|_{BMO_\alpha^2} + c]} \le \|\theta\|_{BMO_\alpha^2} \le c\sqrt{2S + 2k\varepsilon_2 + (c_3U^{0.572} + c_4)U^{1.428}\varepsilon_3 + \varepsilon_3^2}$$

From the formula (2.15), the consequence of the bound of upper and lower bound for $\|\theta\|_{BMO_\alpha^2}$ is that large $\|\theta\|_{BMO_\alpha^2}$ must eventually imply large S, holding the other parameters constants. However, the large S does not necessarily imply large $\|\theta\|_{BMO_\alpha^2}$.

Using the same method, we obtain the upper bound of dispersion coefficient k.

$$k \le [S + \left\|\frac{\partial\theta}{\partial t}\right\|_{BMO_\alpha^2} + 1.5\times (7.428\frac{H}{U_*}U^2 + 1.775H^{0.38}T^{0.620}U_*^{-0.428}U^{1.428} + \|\nabla\theta\|_{BMO_\alpha^2})^2] \div \|\Delta\theta\|_{BMO_\alpha^2} \quad (2.16)$$

If θ is time-independent and satisfies the formula (2.16), then we have

$$k \le [S + 1.5\times (7.428\frac{H}{U_*}U^2 + 1.775H^{0.38}T^{0.620}U_*^{-0.428}U^{1.428} + \|\nabla\theta\|_{BMO_\alpha^2})^2] \div \|\Delta\theta\|_{BMO_\alpha^2}.$$

By the formula (2.9) and (2.16), we obtain the upper and lower bound of the dispersion coefficient k.

$$\frac{S}{\|\theta\|_{BMO_\alpha^2}\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1}\|\Delta\psi\|_{BMO_\alpha^2}} - \frac{\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1}\left\|\frac{\partial\psi}{\partial t}\right\|_{BMO_\alpha^2}}{\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1}\|\Delta\psi\|_{BMO_\alpha^2}} - \frac{7.428H\|u\|_{BMO_\alpha^2}^2}{\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1}\|\Delta\psi\|_{BMO_\alpha^2}U_*}\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1}\|\nabla\psi\|_{BMO_\alpha^2} - \frac{1.775H^{0.38}T^{0.620}\|u\|_{BMO_\alpha^2}^{1.428}}{\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1}\|\Delta\psi\|_{BMO_\alpha^2}(U_*)^{0.428}}\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1}\|\nabla\psi\|_{BMO_\alpha^2}$$
$$\le k \le [S + \left\|\frac{\partial\theta}{\partial t}\right\|_{BMO_\alpha^2} + 1.5\times (7.428\frac{H}{U_*}U^2 + 1.775H^{0.38}T^{0.620}U_*^{-0.428}U^{1.428} + \|\nabla\theta\|_{BMO_\alpha^2})^2] \div \|\Delta\theta\|_{BMO_\alpha^2} \quad (2.17)$$

By the formula (2.17), the increment of U should decrease the lower bound of dispersion coefficient and increase the upper bound of dispersion coefficient, holding the other parameters constants. While increasing S should augment the upper and lower bound of dispersion coefficient, holding the other parameters constants. Conversely, the increment of the upper and lower bound of dispersion coefficient does not imply the increment of source amplitude or the decrement of stirring amplitude. However, the upper and lower bound of dispersion coefficient with large amplitude almost produce the large source amplitude S, unless U is decreased sufficiently.

As a physically meaningful measure of mixing efficiency, the quivalent dispersion coefficient k_{eq} has been investigated in (Young 1999; Pope 2000). Let θ_0 be the solution of the advection dispersion equation with the same source term but no stirring term, then we define

$$\frac{k_{eq}^2}{k^2} = \frac{\|\theta_0\|_{BMO_\alpha^2}^2}{\|\theta\|_{BMO_\alpha^2}^2} \quad (2.18)$$

By the formula (2.17) and (2.18), it is easy to obtain the upper and lower bound of the equivalent dispersion coefficient k_{eq}^2.

$$[\frac{S}{\|\theta\|_{BMO_\alpha^2}\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1}\|\Delta\psi\|_{BMO_\alpha^2}}-\frac{\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1}\left\|\frac{\partial\psi}{\partial t}\right\|_{BMO_\alpha^2}}{\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1}\|\Delta\psi\|_{BMO_\alpha^2}}$$

$$-\frac{7.428H\|u\|_{BMO_\alpha^2}^2}{\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1}\|\Delta\psi\|_{BMO_\alpha^2}U_*}\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1}\|\nabla\psi\|_{BMO_\alpha^2}$$

$$-\frac{1.775H^{0.38}T^{0.620}\|u\|_{BMO_\alpha^2}^{1.428}}{\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1}\|\Delta\psi\|_{BMO_\alpha^2}(U_*)^{0.428}}\cdot$$

$$\sup\limits_{\|\psi\|_{BMO_\alpha^2}=1}\|\nabla\psi\|_{BMO_\alpha^2}]^2\cdot\frac{\|\theta_0\|_{BMO_\alpha^2}^2}{\|\theta\|_{BMO_\alpha^2}^2}\le k_{eq}^2 \tag{2.19}$$

$$\le\frac{\|\theta_0\|_{BMO_\alpha^2}^2}{\|\theta\|_{BMO_\alpha^2}^2}[S+\left\|\frac{\partial\theta}{\partial t}\right\|_{BMO_\alpha^2}+1.5\times$$

$$(7.428\frac{H}{U_*}U^2+1.775H^{0.38}T^{0.620}U_*^{-0.428}U^{1.428}$$

$$+\|\nabla\theta\|_{BMO_\alpha^2})^2]^2\div\|\Delta\theta\|_{BMO_\alpha^2}^2$$

The worst lower bound for the mixing efficiency would be achieved by injecting scalar variance at scale l while stirring to keep the dominant scale of the concentration fluctuation field as L.

Conclusions

By the same method, it is easy to obtain the upper and lower bound for the other parameter of the advection dispersion equation and at the same time we may find the relation among the parameters. It is clear that our results are established on the unit ball B_n, however, our results are obviously correct for the whole research area with any bounded domain Ω. It is assumed that θ_i is the concentration in the ith unit ball and k_i is the dispersion coefficient in the ith unit ball, so

$$\theta=\frac{\theta_1+\theta_2+\cdots+\theta_n}{n}$$

is the concentration of the whole research area and

$$k=\frac{k_1+k_2+\cdots+k_n}{n}$$

is the concentration of the whole research area. Using the same method, the upper and lower bound are obtained for the concentration and dispersion coefficient for the whole research area, as well as the upper and lower bound of equivalent dispersion coefficient for the whole research area.

The theoretical results obtained in this paper will be applied to the realistic examples, for example, to the research for the transfer of dissolved heavy metal in river. According to the formula (2.15), the following parameters are required: top width of the channel; wetted cross-sectional area; the average concentration of dissolved heavy metal for the research area; average depth over the cross section; cross section average velocity; local shear velocity and the selection and calculation of ψ. It is clear that most of these parameters are easy to be obtained by adopting general method, so new way to research the concentration for the dissolved heavy metal is available.

The investigation of these works is left for further research.

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